Fiscal Cooperation and the Permission to Tax

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We analyze the choice of tax bases and rates in a two-stage extensive form game. Our underlying economic model has two regions, an agent in each region who owns the fixed factors there, mobile workers, endogenous incomes, and local public goods that are valuable to both owners and workers. We focus on the case in which the owners collectively determine each region's tax bases if they both agree, otherwise the bases are determined by the workers. Subgame perfect equilibria exist in which the owners do agree in the first stage and then choose tax rates that make *both* regions less attractive to workers than would occur if they had disagreed. The owners in effect use the first stage to reduce tax rate competition in the second stage, a phenomenon we call fiscal cooperation. This result is somewhat surprising given the incentive owners face to compete for the mobile factor. Indeed, fiscal cooperation need not occur if the owners are too aggressive in the first stage. If each owner attempts to deny the opposing region the ability to fund local public goods in hopes of making that region unattractive to workers, they cannot agree in the first stage and the workers determine the bases.

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1. Introduction

Over the past twenty years, economists have developed a tractable model of tax competition. In this model, jurisdictions ("regions") choose tax rates on bases that are prespecified by the analyst. But what if the tax bases are determined endogenously? Would tax rate competition simply become tax base competition, or would some form of fiscal cooperation emerge?

It will be useful to consider a rough example before discussing the general framework. In the United States, all powers of local government come from state constitutions and statutes.[1] Local governments need not, however, be mere agents of the states. Localities as a group have an important voice in determining state statutes through their representatives in the state legislature. What is important here is that it is the citizens of the *entire* state, through a state political process, who make general rules under which localities function. The citizens of each locality, through a local political process, then make meaningful choices within these constraints. In particular, localities receive the permission to tax through state constitutions and statutes. This permission gives localities a meaningful, but hardly unlimited, set of bases for which they then choose rates.

The model we develop captures a few basic features of this environment. Regions share a single political link through the collective choice of tax bases for each region. They share a

[1] "State legislatures have plenary power over local governments," Mandelker et al. (1990), p. 85. As the U.S. Supreme Court wrote in a landmark case, "The number, nature and duration of the powers conferred upon [municipal corporations] and the territory over which they shall be exercised rests in the absolute discretion of the State," Hunter v. City of Pittsburgh, 207 U.S. 161, 178 (1907). State legislation over municipalities must, of course, be consistent with the state's own constitution and the federal constitution.

single economic link through the presence of a mobile factor. No other interactions, like fiscal federalism, are permitted. Although this is a highly stylized environment, it does provide a number of insights into the incentives created by political interdependence and the effect it can have on regional economies. This is of course an important step in analyzing the incentive to form such a political linkage in the first place. Our focus, however, is primarily on the determinants of the permission to tax in each region.

It is important to note at the outset that we do not assume that all regions must be granted the same permission to tax. To the contrary, the permission to tax is granted to each region individually, or "by name." This is the most interesting environment in which to explore the possibility of fiscal cooperation. Each region can attempt to constrain the opposing region without constraining itself. This in itself would tend to make cooperation unlikely. It is also true that, both historically and today, there is significant variation in the powers of local governments.[2]

The economic model on which we build is a form of the "regional" Tiebout model (Mieszkowski and Zodrow (1989)). First, there are fixed and mobile factors (land and labor), and the migration of mobile factors has different direct effects on the incomes of residents

[2]The form in which local authority is granted and the extent of direct state involvement in local affairs has changed over time. Through the late nineteenth century both state constitutions and state statutes frequently addressed particular localities by name. Burns and Gamm (1997) provide a detailed empirical study of 6415 bills from three state legislature in the period 1871-1921. They conclude, "the ordinary work of state politics was local affairs, and an ordinary branch of local government was the state legislature (p. 19)."

Modern state constitutions typically define a few classes of municipalities according to property value or population. They then require the powers of each municipality in the same class to be the same. This has reduced legislation that targets particular localities by name, but it has not eliminated statutes that are *in effect* about a single jurisdiction. For example, in Missouri it has been reported that the state legislature has in effect created 167 classifications of cities, despite the fact that the state constitution permits at most four (Dohm (1985), p. 198).

depending on their type (owner or worker). Migration creates winners and losers in partial equilibrium, and we want the model to capture this basic conflict. Second, agents derive utility from a local public good and a private good. This provides all residents with a direct common interest that is absent in models of pure redistribution, but does not prevent redistribution from taking place. Third, migration to or from any single region affects all regions and is anticipated. Regions therefore engage in strategic interaction, which is consistent with a context in which there are just a few regions and they are not wildly disproportionate in size. Finally, government action at all levels is determined through a simple collective choice process that is consistent with majority rule.

Given this economic model, we specify a two-stage extensive form game called the *tax structure game*. In the first stage, if all of the owners propose the same national tax structure, then it wins. A national tax structure specifies the permission that all regions have to tax wages, rents, both kinds of income, or neither. If the owners make different proposals then the workers' proposal wins. In the second stage, the owners in each region choose tax rates for the bases that are permitted in their regions. After this, workers are assigned to regions so that they are in migration equilibrium. This essentially requires that no individual worker has an incentive to migrate.

The central findings of the paper are as follows. There are equilibria to the tax structure game in which all of the owners propose the same national tax structure. They therefore implicitly cooperate. There are also equilibria in which the owners make different proposals. The latter equilibria help to explain the former. Equilibria with cooperation are somewhat surprising given the incentive the owners have to compete for the mobile factor. Cooperation, however, has two benefits. First, it allows the owners to reduce tax competition in the second stage. That is to say, it allows them to choose tax rates that make *both* regions less attractive to workers than would occur without cooperation. Second, cooperation keeps the choice of tax structure out of the hands of the workers. If the workers determine tax structure then the outcome for the owners is poor, even though the owners still determine tax rates in the second stage. For the owners to fail to cooperate, then, they must each be so aggressive in the first stage and propose restrictions so severe on the other region that each prefers to let the workers determine the national tax structure. Thus, the owners cooperate as long as each one does not try to gain too much.

There is of course a large literature using the regional Tiebout model. Hoyt (1991) considers the choice of exogenously specified tax instruments in models with mobile factors and anticipated migration. A prisoner's dilemma governs the choice of tax instruments and pushes all communities to use the property tax instead of the income tax. Our results are consistent with Hoyt's, since the first stage of our game gives the players an opportunity to create a binding agreement to constrain how they play in the second stage. Henderson (1994, 1995) presents related ideas.

More recently, Nechyba (1997) develops a general equilibrium model of the two-tiered public sector that incorporates a rich set of endogenous variables and considers all existence issues. Like Hoyt, Nechyba shows that a prisoner's dilemma governs community choice of the property tax versus the income tax. However, behavior in his model does not occur in stages. Fiscal variables and regional populations are determined simultaneously. Migration is therefore not anticipated in the sense used in this paper. In contrast, Hindricks (2001) and Braid (2005) do develop two-stage models. Hindricks focuses on pure redistribution and does not include local public goods in his model. Braid does include local public goods, but he holds the number of workers in each region fixed. He therefore cannot give local public goods a role in attracting workers to a region.

Section 2 presents the economic model and section 3 presents the tax structure game. Section 4 gives preliminary results and section 5 gives the main results and a discussion. Section 6 concludes.

2. The Model

We assume there are just two regions, denoted A and B. There are two general classes of agents, owners (she) and workers (he). They consume private good and a pure local public good.

Each of the $\hat{n} \geq 2$ owners resides in a single exogenously specified region. The number of owners in region i is denoted by the integer $\hat{n}_i$, and their income comes from ownership of an equal share of land in that region. The pair $(\hat{n}_A, \hat{n}_B)$ denotes the distribution of owners across regions and satisfies:

$$(\hat{n}_A, \hat{n}_B) \in \hat{\mathcal{N}} \equiv \{(\hat{n}_A, \hat{n}_B) \mid \hat{n}_A + \hat{n}_B = \hat{n},\ \hat{n}_A \geq 1,\ \hat{n}_B \geq 1\} \tag{1}$$

Each of the $n \geq 1$ workers is endowed with a single unit of labor. Each one works and consumes in a single region and is free to locate in either region. We set aside the integer problem by allowing fractions in each location.[3] The pair (n_A, n_B) denotes the distribution of workers working and consuming in each region and satisfies:

$$(n_A, n_B) \in \mathcal{N} \equiv \{(n_A, n_B) \mid n_A + n_B = n,\ n_A \geq 0,\ n_B \geq 0\} \tag{2}$$

In region i, fixed factors and mobile labor produce an all purpose good with technology:

$$f_i(n_i) = \phi_i n_i^{\beta}, \qquad \phi_i > 0,\ 0 < \beta < 1 \tag{3}$$

Owners hire workers and pay them their marginal product, which is then taxed at rate $w_i \in [0,1]$. Private good consumption for a worker is therefore $c_{iw} \equiv (1 - w_i) f_i'$. Owners receive the revenue that remains, which is then taxed at rate $r_i \in [0,1]$. Private good consumption by an owner is $c_{ir} \equiv (1 - r_i)(f_i - n_i f_i')/\hat{n}$. The all purpose good can be tranformed into private good and local public good, z_i, at a constant rate, so $z_i = w_i n_i f_i' + r_i(f_i - n_i f_i')$.

Preferences for all agents take the form $U(c_{ij}, z_i) = c_{ij}^{\alpha} z_i,\ j = w, r,\ i = A, B$, with $0 < \alpha < \infty$. Conditional on a quantity of labor in region i, the preferences of owners over tax rates may be derived by substituting c_{ir} and z_i into the utility function. This gives:

$$\hat{V}_i(w_i, r_i, n_i) \equiv H_i(w_i, r_i) n_i^{\beta(\alpha+1)}, \ \ i = A, B \tag{4}$$

where $H_i(w_i, r_i) = (1 - r_i)^{\alpha}[\beta w_i + (1 - \beta) r_i] h_i,\ h_i = \phi_i^{\alpha+1}[(1 - \beta)/\hat{n}_i]^{\alpha}$. For reasons that become clear further on, we assume that these preferences are concave in n_i:

$$\beta(\alpha + 1) - 1 < 0 \tag{5}$$

[3] In earlier work, we formally solved the integer problem by allowing a single individual to commute between regions. We lose nothing important by simplifying the analysis. Assuming a continuum of workers would introduce conceptual problems given the presence of a pure (or even somewhat impure) public good; see Berliant and Rothstein (1998).

Figure 1 graphs indifference curves of $\hat{V}_A$ for $n_A > 0$. The maximum on the unit square occurs at $\hat{w}_A^* = 1$ and $\hat{r}_A^* = \frac{1-\beta(1+\alpha)}{(1-\beta)(1+\alpha)}$.

Figure 1

In the tax structure game we study below, a representative owner in each region will choose rates for the taxes that the region has permission to use, taking into account the migration response of workers. Thus, we need to examine how tax rates influence the incentive to migrate. We define the utility a worker in region $-i$ expects to achieve after migrating to region i to be the utility of worker residents in i, if there are any, or the naturally defined limit utility otherwise.[4] Formally:

$$V_i[G_i(w_i, r_i), n_i] \equiv U(c_{iw}, z_i) = G_i(w_i, r_i) \left(\frac{1}{n_i}\right)^{\alpha-\beta(1+\alpha)} \qquad \text{if} \quad n_i > 0 \tag{6}$$

$$V_i[G_i(w_i, r_i), 0] \equiv \lim_{\xi \downarrow 0} G_i(w_i, r_i) \left(\frac{1}{\xi}\right)^{\alpha-\beta(1+\alpha)} = \begin{cases} +\infty & \text{if} \quad G_i(w_i, r_i) > 0 \\ 0 & \text{if} \quad G_i(w_i, r_i) = 0 \end{cases} \tag{7}$$

where:

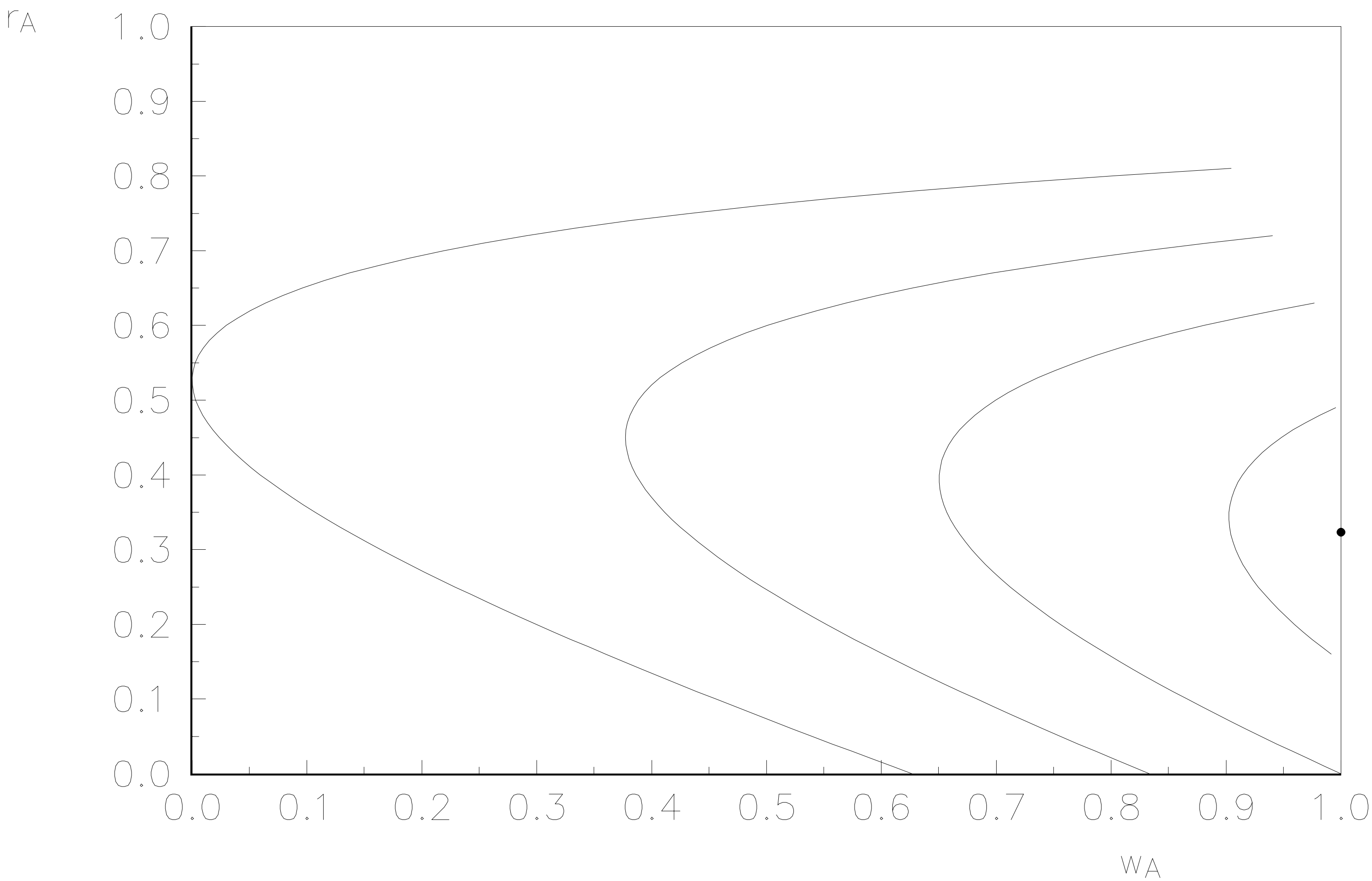
$$G_i(w_i, r_i) = (1 - w_i)^\alpha[\beta w_i + (1 - \beta)r_i]g_i, \quad g_i = \phi_i^{\alpha+1}\beta^\alpha$$

G_i plays a fundamental role in the model and analysis, for reasons that become clear below.

If the exponent in (6) is positive, then the utility of existing residents decreases with population. This condition is required for reasonable comparative statics (it is the "stability"

[4]Strictly speaking, there should be no difference between what an agent "expects" to receive and what he would receive, and the latter is already defined in all cases. Our approach provides a simple and intuitive resolution to integer problems and empty region problems, keeps the model analytically tractable, and allows us to focus on other issues.

Figure 1



condition for this model). We therefore impose:[5]

$$\alpha - \beta(1+\alpha) > 0 \tag{8}$$

Regarding equation (7), we have $G_i(w_i, r_i) = 0$ if and only if the taxes in region i leave a worker with no private good ($w_i = 1$) or fund no public good ($w_i = 0$ and $r_i = 0$). These tax rates are, in an intuitive sense, extreme. Denote the set of *extreme tax rates* in region i by:

$$\mathcal{E}_i = \mathcal{E}_i^0 \cup \mathcal{E}_i^1$$

where:

$$\mathcal{E}_i^0 = \{(0,0)\}, \quad i = A, B$$

$$\mathcal{E}_i^1 = \{(w_i, r_i) | w_i = 1, 0 \leq r_i \leq 1\}, \quad i = A, B$$

It now follows that:

$$V_i[G_i(w_i, r_i), 0] = \begin{cases} +\infty & \text{if} \quad (w_i, r_i) \in [0,1]^2 \setminus \mathcal{E}_i \\ 0 & \text{if} \quad (w_i, r_i) \in \mathcal{E}_i \end{cases} \tag{9}$$

As for the interpretation, $V_i[G_i(w_i, r_i), 0] = +\infty$ when tax rates are moderate ("non-extreme") simply means there is always an incentive for migration into an unoccupied region with moderate tax rates. On the other hand, $V_i[G_i(w_i, r_i), 0] = 0$ when tax rates are extreme means there is never an incentive for migration into an unoccupied region with extreme rates. This

[5]Note that (5) and (8) are together equivalent to:

$$\beta < \min\left\{\frac{\alpha}{\alpha+1}, \frac{1}{\alpha+1}\right\}$$

is reasonable, since a worker who migrated to region i would find himself lacking either the private good or the public good (or both).

Figure 2 graphs indifference curves of V_A for $n_A > 0$ and indicates the extreme rates. Notice that V_A is maximized with $w_A^* = 0$ and $r_A^* = 1$. While the owners would tax themselves even after taking all of the income from workers, the workers would not do the same after taking all of the income from owners. This results from (8), which implies that α is large enough so that the taste for the local public good is not "too" strong.[6]

Figure 2

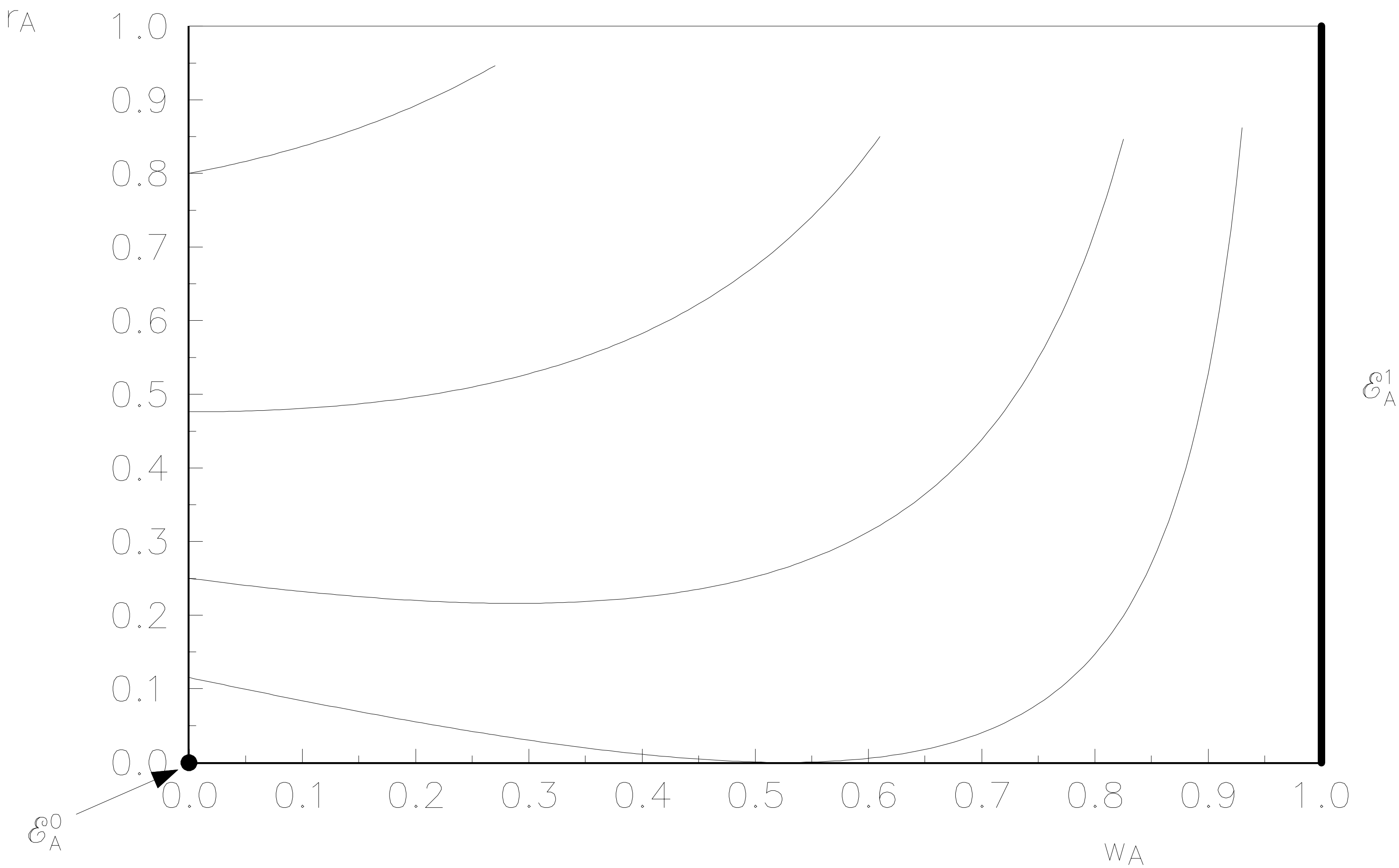
Given any vector of tax rates $(w_A, r_A, w_B, r_B) \in [0,1]^4$, a *migration equilibrium* is any allocation of workers (n_A, n_B) such that $n_i \geq 0$ for $i = A, B$, $n_A + n_B = n$, and:

$$n_i > 0 \Rightarrow V_i[G_i(w_i, r_i), n_i] \geq V_{-i}[G_{-i}(w_{-i}, r_{-i}), n_{-i}], \; i = A, B \tag{10}$$

Our first result is:

[6]Indeed, workers would prefer to use some of the tax revenue from the owners to subsidize their own consumption of private good. Given $r_A = 1$, workers would choose $w_A = 1 - \frac{\alpha}{\beta(1+\alpha)}$, which is negative given (8).

Figure 2



r_A
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
w_A
$\mathcal{E}_A^0$
$\mathcal{E}_A^1$

Theorem 1.

i. Given any vector of tax rates $(w_A, r_A, w_B, r_B) \in [0,1]^4$, the allocation (N_A, N_B) defined by the following functions is a migration equilibrium:[7]

$$N_A(w_A, r_A, w_B, r_B) =$$

$$\begin{cases} \frac{n[G_A(w_A,r_A)]^\gamma}{[G_A(w_A,r_A)]^\gamma+[G_B(w_B,r_B)]^\gamma} & \text{if } (w_A, r_A, w_B, r_B) \in [0,1]^4 \setminus \mathcal{E}_A \times \mathcal{E}_B \\ n^{00} & \text{if } (w_A, r_A) \in \mathcal{E}_A^0 \text{ and } (w_B, r_B) \in \mathcal{E}_B^0 \\ n^{01} & \text{if } (w_A, r_A) \in \mathcal{E}_A^0 \text{ and } (w_B, r_B) \in \mathcal{E}_B^1 \\ n^{10} & \text{if } (w_A, r_A) \in \mathcal{E}_A^1 \text{ and } (w_B, r_B) \in \mathcal{E}_B^0 \\ n^{11} & \text{if } (w_A, r_A) \in \mathcal{E}_A^1 \text{ and } (w_B, r_B) \in \mathcal{E}_B^1 \end{cases}$$

$$N_B(w_A, r_A, w_B, r_B) = n - N_A(w_A, r_A, w_B, r_B)$$

and $0 \leq n^{jk} \leq n$, $j = 0, 1$; $k = 0, 1$; and $\gamma \equiv \frac{1}{\alpha - \beta(1+\alpha)}$.[8]

ii. The migration equilibrium is unique if $(w_A, r_A, w_B, r_B) \in [0,1]^4 \setminus \mathcal{E}_A \times \mathcal{E}_B$.

iii. $N_i(w_A, r_A, w_B, r_B)$, $i = A, B$, is discontinuous on $\mathcal{E}_A \times \mathcal{E}_B$ and continuous elsewhere.

The *payoff function for owner i* is the function $\hat{\mathcal{V}}_i : [0,1]^4 \to [0, \infty)$ representing the preferences over tax rates of landowners in region i, taking into account migration equilibrium. It comes form substituting N_i into $\hat{V}_i$:

$$\hat{\mathcal{V}}_i(w_A, r_A, w_B, r_B) = H_i(w_i, r_i)[N_i(w_A, r_A, w_B, r_B)]^{\beta(\alpha+1)}, \quad i = A, B \tag{11}$$

[7]We avoid the notation $(w_i, r_i, w_{-i}, r_{-i})$ when it could be confusing.

[8]We have $\gamma > 0$ by (8). One should also note that $[0,1]^4 \setminus \mathcal{E}_A \times \mathcal{E}_B \neq ([0,1]^2 \setminus \mathcal{E}_A) \times ([0,1]^2 \setminus \mathcal{E}_B)$. For example, the vector $(w_A, r_A, w_B, r_B) \in [0,1]^4 \setminus \mathcal{E}_A \times \mathcal{E}_B$ could have $(w_A, r_A) \in \mathcal{E}_A$ or $(w_B, r_B) \in \mathcal{E}_B$, although not both.

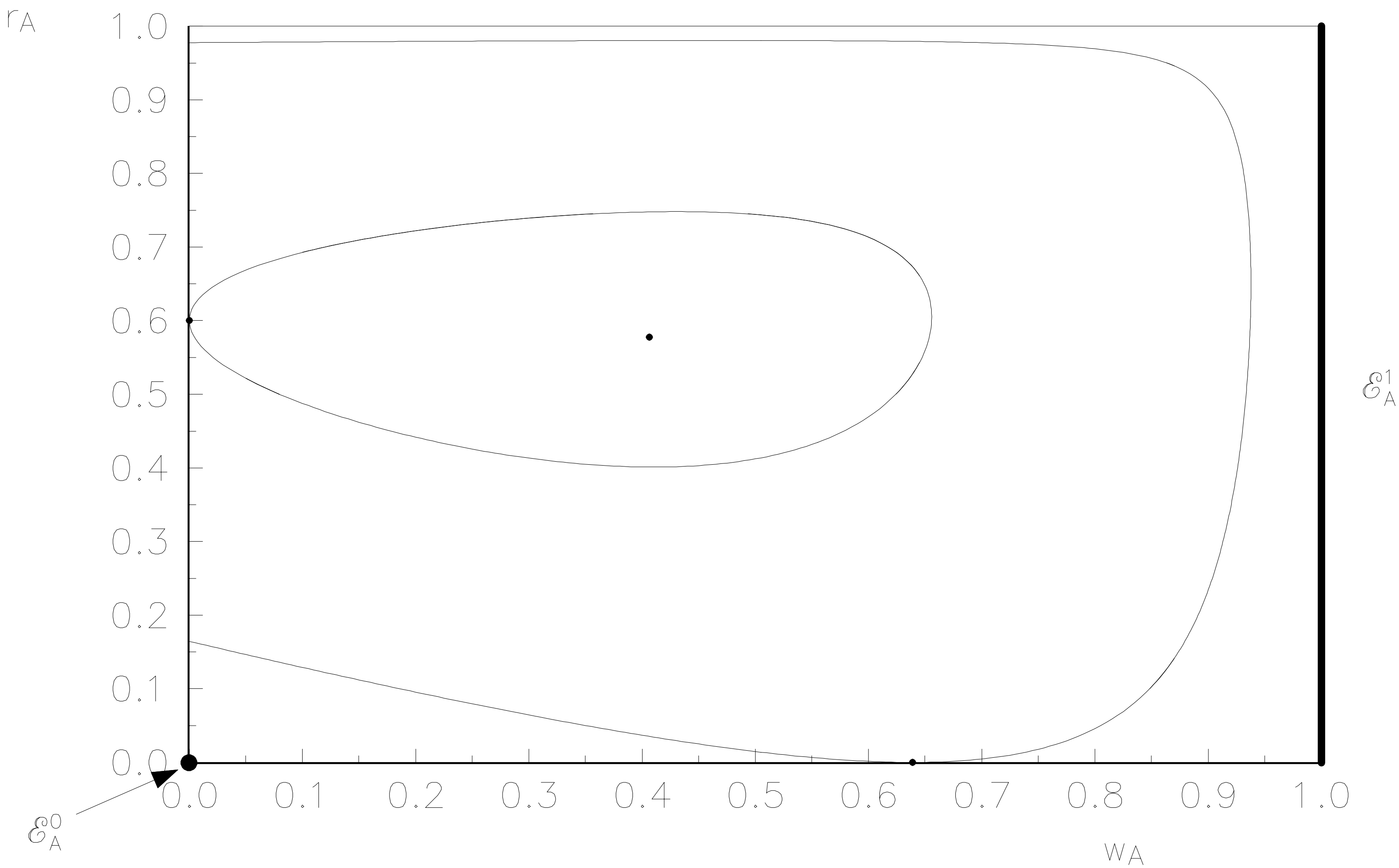
Figure 3 presents indifference curves of payoffs for owner A given fixed non-extreme tax rates in region B. Owners have preferences over taxes in both regions because of worker migration. If owners can choose both tax rates then they would choose the "ideal point." If they are constrained to tax rents only then they would choose the tangency with the vertical axis. If they are constrained to tax wages only then they would choose the tangency with the horizontal axis.[9]

Figure 3

It is clear from (11) that $\hat{\mathcal{V}}_i$ is necessarily continuous wherever N_i is continuous. $\hat{\mathcal{V}}_i$ is discontinuous at any point where N_i is discontinuous provided $H_i(w_i, r_i) > 0$, otherwise it is continuous there. Intuitively, a discontinuous shift in the quantity of mobile factor in any region will cause the incomes of the owners of the fixed factors there to jump. This will cause their payoffs to jump unless taxes are such that the extra income does them no good, which is to say $H_i(w_i, r_i) = 0$. The latter occurs if and only if $r_i = 1$, so all their income is taxed away (leaving them no private good) or $w_i = 0$ and $r_i = 0$, so none of their income is taxed away (giving them no local public good). If N_i is discontinuous then the taxes in both regions must be extreme. If in addition $H_i(w_i, r_i) = 0$ then the taxes must be either

[9] Given any moderate (w_A, r_A) and holding the number of workers constant, an increase in w_A leads to an increase in the quantity of local public good. This directly benefits owners. Furthermore, if the tax on rents is low enough, then the increase in the wage tax increases $G_i(w_A, r_A)$ and causes in-migration of workers. Both effects tend to increase owner utility. However, once the wage tax is high enough, the migration flow reverses. Eventually the cost to owners from out-migration outweighs the benefit of placing the nominal tax on workers and owner utility decreases with the tax. Thus, as we move left to right in Figure 3, owner payoffs increase and then decrease.

Figure 3



r_A
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0.0
0.1
0.2
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0.4
0.5
0.6
0.7
0.8
0.9
1.0
w_A
$\mathcal{E}_A^1$
$\mathcal{E}_A^0$

$(0,0)$ and $(1,1)$ (see Figure 2). These are the extreme points at which we expect the owner payoff function to be discontinuous but it is not.

We therefore have the following. Define the set:

$$\overline{\mathcal{E}} = \{(0,0,0,0),(1,1,0,0),(0,0,1,1),(1,1,1,1)\}$$

Theorem 2.

i. Either $\hat{\mathcal{V}}_A$ or $\hat{\mathcal{V}}_B$ (or both) is discontinuous at any point in $(\mathcal{E}_A \times \mathcal{E}_B) \setminus \overline{\mathcal{E}}$.

ii. Both owner payoff functions are continuous on the complement of this set in $[0,1]^4$, i.e., on $\overline{\mathcal{E}} \cup ([0,1]^4 \setminus \mathcal{E}_A \times \mathcal{E}_B)$.

Thus, this model contains a continuum of points at which one or both owner payoff functions are discontinuous. Furthermore, at some of these points, an owner's payoff function fails to be upper semi-continuous in her own strategy. This has important implications for the existence of best replies and Nash equilibrium. We return to this issue in section 3 below.

The *worker payoff function* (or *common worker utility*) is the function $\mathcal{V} : [0,1]^4 \to [0,\infty)$ representing the preferences of workers over tax rates taking into account migration equilibrium. Given any $(w_A, r_A, w_B, r_B) \in [0,1]^4$, it follows from substituting the implied value of N_i from Theorem 1 and the pair (w_i, r_i) into (6) or (7) as appropriate. In all cases:

$$\mathcal{V}(w_A, r_A, w_B, r_B) = \left(\frac{1}{n}\right)^{\frac{1}{\gamma}} \{[G_A(w_A, r_A)]^\gamma + [G_B(w_B, r_B)]^\gamma\}^{\frac{1}{\gamma}} \tag{12}$$

Common worker utility is everywhere continuous, even at points where the number of workers in a region is not.

Any change in taxes that increases G_i necessarily increases the number of workers in region i (Theorem 1). It also increases all workers' post-migration utility (equation (12)). G_i is, in an intuitive sense, an index of the attractiveness of region i to workers. We use this idea in the definition of fiscal cooperation below.

3. The Tax Structure Game

We now develop a 3-player 2-stage extensive form game to model the determination of tax structure and rates. The players in the game are arbitrary owners from regions A and B and an arbitrary worker. Owners in the same region and workers in both regions will receive the same payoffs, so we lose none of the interesting incentives by letting one individual serve as a representative for the group.

Any of the four factor taxes defined above may be denoted j_i, where $j = w, r$ and $i = A, B$. We say that region i has the *permission to tax* factor j if it is possible to have $j_i > 0$, i.e., if the tax rate on factor j need not be zero. Define the sets P_{j_i} to be either $[0, 1]$ or $\{0\}$. Assume also that we must have $j_i \in P_{j_i}$; we impose this here for ease of exposition, but it is an explicit part of the extensive form defined below. Then region i has the permission to tax factor j if and only if $P_{j_i} = [0, 1]$. Equivalently, the permission is denied if and only if $P_{j_i} = \{0\}$.

Define a *tax structure for region* i to be the ordered pair $p_i = (P_{w_i}, P_{r_i})$. Thus, $p_i =$

$([0,1],\{0\})$ means that region i has permission to use only the wage tax. This notation can be improved upon by defining the following equivalences:

$$wr \equiv ([0,1],[0,1]), \quad w0 \equiv ([0,1],\{0\}), \quad 0r \equiv (\{0\},[0,1]), \quad 00 \equiv (\{0\},\{0\})$$

Now the expression $p_i = w0$ also states that region i has permission to use only the wage tax. More generally, we have:

$$p_i \in \{wr, w0, 0r, 00\}$$

Define a *national tax structure* to be the ordered pair $p = (p_A, p_B)$. For example, $p = (w0, wr) = (([0,1],\{0\}),([0,1],[0,1]))$ indicates that region A has permission to use the wage tax only and region B has permission to use both taxes. Again, a better yet equally unambiguous notation is $p_A p_B$ instead of (p_A, p_B). This gives $w0wr$ instead of $(w0, wr)$.

Note that there are sixteen distinct national tax structures. In nine of them, both regions have the permission to tax (i.e., use one or both taxes). In six of the remaining seven only one region has the permission to tax. In the last case neither region has the permission to tax.

The stages of play are as follows. In the first stage of the game, each player simultaneously proposes a national tax structure $p = p_A p_B$. If the owners propose the same national tax structure then it is adopted. If they make different proposals then the worker's proposal is adopted.[10]

[10]This result is implied by the following more general rule. Weight the worker's proposal by n and owner i's proposal by $\hat{n}_i$. Assume $\hat{n}_A + \hat{n}_B > n$, so there are more owners in total, but $n > \max\{\hat{n}_A, \hat{n}_B\}$, so there are more workers in total than owners in each region. Let the proposal with the greatest weight win.

In the second stage, each owner unilaterally chooses tax rates $(w_i, r_i) \in p_i$ where p_i is part of the winning national tax structure. In other words, each owner chooses tax rates in her region that are consistent with the winning national tax structure.[11] The winning national tax structure therefore specifies the factors that each region has permission to tax, as defined above.

The assumption that the owners of fixed factors choose tax rates in the second stage is most consistent with the fiscal competition literature (Zodrow and Mieszkowski (1986), Wildasin (1991)). It puts the owners in direct competition for the mobile factor in the second stage. This is also the most interesting context in which to study the incentive for the owners to cooperate in determining the national tax structure in the first stage.

After the second stage, workers are assigned locations so that the totals in each region form a migration equilibrium. In other words, a worker does not actually choose a location, he is assigned a location as part of the rules of the game. The assignment has the property that no individual worker could achieve a higher payoff by migrating. This is a useful and reasonable simplification given that the migration equilibrium is unique unless the tax rates are extreme in both regions (Theorem 1 (*ii*)). Payoffs are then realized.

A *strategy* for player i, denoted s_i, is a choice of national tax structure; and if the player is an owner, a choice of tax rates in her region conditional on the winning national tax

[11] This result is formally implied by the following. Assume that the owners in each region propose tax rates for their respective regions and the worker proposes tax rates for both regions. Weight the worker's proposals by n_A^0 and n_B^0, where these satisfy $n_i^0 \geq 0$ and $n_A^0 + n_B^0 = n$. The distribution (n_A^0, n_B^0) is essentially a pre-migration or *ex ante* distribution of regional political rights for workers. Weight the owners' proposals by $\hat{n}_A$ and $\hat{n}_B$. Assume $\hat{n}_A > n_A^0$ and $\hat{n}_B > n_B^0$ (note that there is no inconsistency with the previous assumptions). Let the proposals with the greatest weight in each region win.

structure and each player's proposal for a national tax structure. A *strategy profile*, denoted s, is a triple consisting of a strategy for the worker, the owner in A, and the owner in B.

In summary, the *tax structure game* consists of the three players {worker, owner in A, owner in B}, the two-stage extensive form above, and the payoff functions in (11) and (12). Note that there are four parameters, $\phi_A, \phi_B, \alpha, \beta$. The restrictions above imply:

$$\left\{(\phi_A, \phi_B, \alpha, \beta) \mid \phi_A > 0,\ \phi_B > 0,\ 0 < \alpha < \infty,\ 0 < \beta < \min\left\{\frac{\alpha}{\alpha+1}, \frac{1}{\alpha+1}\right\}\right\} \tag{13}$$

4. Second-Stage Nash Equilibria

We begin by analyzing the Nash equilibrium tax rates and payoffs on each of the subgames defined by the sixteen national tax structures.[12] In nine cases, both regions have some permission to tax. In six cases, only one region has permission to tax. A Nash equilibrium consists of the tax rates that unilaterally maximize that region's payoff function given zero tax rates for the other region. In the last case, the tax rates are all zero.

One question that immediately arises is whether a second-stage Nash equilibrium exists in all cases. Existence is problematic because of the discontinuities in owner payoff functions that we established in Theorem 2. The answer is positive:[13]

[12] Strictly speaking, subgames are defined by a national tax structure and the proposals made by each player in the first stage, not just a national tax structure. The rationale for ignoring the proposals in our analysis here is that it has no effect on the subgame perfect equilibria of the original game. The payoff functions depend only on the winning tax structure and not on this other information.

[13] Theorem 3 uses the existence result in Rothstein (2004) for the nine cases in which both regions have some permission to tax. The remaining cases are more straightforward, but they require the additional restrictions $n^{01} = 0$ and $n^{10} = n$ stated in the premise.

Theorem 3. A second-stage Nash equilibrium exists for all sixteen national tax structures provided $n^{01} = 0$ and $n^{10} = n$.

We now partially characterize the equilibrium tax rates and payoffs. Fix the following notation. Given a national tax structure $p_A p_B$ (so p_A and p_B are in $\{wr, w0, 0r, 00\}$), denote Nash equilibrium tax rates by:

$$(w_A^{p_A p_B}, r_A^{p_A p_B}, w_B^{p_A p_B}, w_B^{p_A p_B})$$

The equilibrium payoff for the worker is:

$$\mathcal{V}^{p_A p_B} \equiv \mathcal{V}(w_A^{p_A p_B}, r_A^{p_A p_B}, w_B^{p_A p_B}, w_B^{p_A p_B})$$

The equilibrium payoff for the owner in region i is:

$$\hat{\mathcal{V}}_i^{p_A p_B} \equiv \hat{\mathcal{V}}_i(w_A^{p_A p_B}, r_A^{p_A p_B}, w_B^{p_A p_B}, w_B^{p_A p_B}), \quad i = A, B$$

Theorems 4 and 5 presents our main characterizations.

Theorem 4. Given a tax structure game and any p_A and p_B in $\{wr, w0, 0r, 00\}$:

i. $$\mathcal{V}^{0r p_B} \geq \mathcal{V}^{p_A p_B}$$

with equality if and only if $p_A = 0r$.

ii. $$\mathcal{V}^{p_A 0r} \geq \mathcal{V}^{p_A p_B}$$

with equality if and only if $p_B = 0r$.

iii. $\mathcal{V}^{0r0r} \geq \mathcal{V}^{p_A p_B}$

with equality if and only if $p_A = p_B = 0r$.

Theorem 5. Given a tax structure game and any p_A and p_B in $\{wr, w0, 0r, 00\}$:

i. $\hat{\mathcal{V}}_i^{0rp_B} \leq \hat{\mathcal{V}}_i^{wrp_B}, \;\; i = A, B$

with equality if and only if $i = B$ and $p_B = 00$.

ii. $\hat{\mathcal{V}}_i^{p_A 0r} \leq \hat{\mathcal{V}}_i^{p_A wr}, \;\; i = A, B$

with equality if and only if $i = A$ and $p_A = 00$.

iii. $\hat{\mathcal{V}}_i^{0r0r} < \hat{\mathcal{V}}_i^{0rwr} < \hat{\mathcal{V}}_i^{wrwr}, \;\; i = A, B$

$\hat{\mathcal{V}}_i^{0r0r} < \hat{\mathcal{V}}_i^{wr0r} < \hat{\mathcal{V}}_i^{wrwr}, \;\; i = A, B$

iv. $0 = \hat{\mathcal{V}}_A^{000r} < \hat{\mathcal{V}}_A^{0r0r} < \hat{\mathcal{V}}_A^{0r00}$

$0 = \hat{\mathcal{V}}_B^{0r00} < \hat{\mathcal{V}}_B^{0r0r} < \hat{\mathcal{V}}_B^{000r}$

The main result in Theorem 4 (part *iii*) states that equilibrium payoffs for workers are highest when the national tax structure is $0r0r$. If the worker could unilaterally choose the national tax structure, he would give both regions the permission to tax only the fixed factors. It is intuitive that the workers should benefit from restricting the owners to taxing just themselves.[14]

Theorem 5, part *iii*, establishes a basic tension over tax structure between each owner and the worker. It says that if an owner could unilaterally choose the national tax structure, she would prefer to give both regions the permission to tax both factors over the permission to tax just the fixed factors. Each owner wants the option to tax the worker and herself and not just herself. The worker has the reverse preference.[15]

The previous result establishes some basis for agreement between the owners. They are still in competition for labor, however. This competition may assert itself in the first stage in efforts by each owner to make the *other* region very unattractive to labor. For example, each owner may try to deny the other region any permission to tax. This would make it impossible for the region to provide any local public good. Whether this incentive is strong enough to disrupt cooperation in the first stage can only be answered by analyzing the equilibria of the overall game. We now turn to this issue.

5. Fiscal Cooperation and Subgame Perfect Equilibria in the Tax Structure Game

We begin by defining fiscal cooperation. Intuitively, this occurs if the owners use the first stage to reduce tax rate competition in the second stage. One requirement, then, is that

[14]Recall, however, that the worker also benefits from the local public goods. How much the result depends on the fact that owners must compete for workers and how much it depends on the fact that owners also benefit from the local public good is an open question.

[15]The owner is also better off if just the *other* owner has this option; we have $\hat{\mathcal{V}}_A^{0r0r} < \hat{\mathcal{V}}_A^{0rwr}$ and $\hat{\mathcal{V}}_B^{0r0r} < \hat{\mathcal{V}}_B^{wr0r}$. This is a more subtle point that we return to in section 5.

the owners propose the same national tax structure. The other requirement is that they then choose tax rates that make *both* regions less attractive to workers than if they had not cooperated.

Formally, given a strategy profile s that forms a subgame perfect Nash equilibrium of the tax structure game, let p denote the winning national tax structure, p' the worker's proposal, and $(w_A, r_A, w_B, r_B) \in p$ and $(w'_A, r'_A, w'_B, r'_B) \in p'$ the Nash equilibrium tax rates on the respective subgames of s. The owners engage in *fiscal cooperation* at s if both owners propose p and $G_i(w_i, r_i) < G_i(w'_i, r'_i)$ for $i = A, B$.

In Theorem 3, we overcame a somewhat difficult existence problem for second-stage Nash equilibria. We face somewhat the opposite problem in the overall game. Any national tax structure p can occur in a subgame perfect equilibrium of the original game if all three players propose it and then play Nash (or choose a unilateral optimum) thereafter. A deviation by any one player would not change the winning tax structure.[16] It follows that the payoff functions are unaffected by the deviation since they depend only on the winning tax structure. Given subgame perfection, the choice of tax rates after the deviation is unchanged and all payoffs are the same. Thus, no agent has an incentive to deviate.

Although any national tax structure can occur in a subgame perfect equilibrium of the original game, it is very misleading to conclude that anything can happen. In Theorem 4, we saw that if the worker could unilaterally choose the national tax structure, he would give both regions the permission to tax the fixed factors only. This strongly suggests that this

[16] If the worker deviates then the two owners are still proposing p, so it wins. If one of the owners deviates then the worker is still proposing p, so it wins.

is the only reasonable national tax structure for him to propose. For him to do otherwise, he must believe that when the tax structure he chooses also wins, that one or both of the owners will choose tax rates that do *not* form a Nash equilibrium (or unilateral optimum) in the second stage. We therefore restrict attention to strategies for the workers in which they choose the national tax structure $0r0r$.

Our central results are as follows. First, fiscal cooperation does occur.

Theorem 6. There exists a strategy profile s that is a subgame perfect Nash equilibrium of the tax structure game and in which the workers propose $0r0r$, both owners propose $0rwr$, and the owners engage in fiscal cooperation at s. The result continues to hold if we replace $0rwr$ with $wr0r$ or with $wrwr$.

Second, fiscal cooperation need not necessarily occur:

Theorem 7. There exists a subgame perfect Nash equilibrium of the tax structure game in which the workers propose $0r0r$, the owners in region A propose $0r00$, and the owners in region B propose $000r$. The winning national tax structure is $0r0r$.

Fiscal cooperation is somewhat surprising given the incentive the owners have to compete for the mobile factor. They also, however, have an incentive to cooperate (or perhaps collude) against the mobile factor. Both owners are better off if they make both regions less attractive to the workers. By agreeing on national tax structure in the first stage, the owners credibly commit to doing just this. Cooperating in the first stage also keeps the choice of tax structure out of the hands of the workers. When this occurs the outcome for the owners is poor, even though the owners still determine tax rates in the second stage.

Cooperation is not assured, though. It fails to occur if the owners are too aggressive in the first stage. In particular, it is possible for each owner to propose tax base restrictions that are so severe on the other region that each prefers to let the workers determine the national tax structure. In this case the owners' proposals differ, neither has an incentive to deviate to the other's proposal, and the worker's proposal wins.

Some insight into the nuances of the model come from closely examining the equilibrium in which both owners propose $0rwr$ and the workers propose $0r0r$. It is intuitive that the owner in B would not deviate to $0r0r$, since then she can only tax her herself. It is less intuitive that the owner in A does not want to deviate. The reason is that forcing the owner in B to only tax himself makes that region more attractive to workers. This forces A to respond, which makes her worse off. Thus, the owners in both A and B benefit when the owner in B taxes both labor and fixed factors.

Although the incentives for fiscal cooperation are strong, both owners may still try to increase their payoffs by making just *the other* region less attractive to workers. As Theorem 7 shows, suppose the owner in A proposes $0r00$ and the owner in B proposes $000r$. Each owner therefore uses the first stage to attempt to deny the other region any ability to tax in the second stage. This would make it impossible for that region to provide any local public good and make it unattractive to the workers. With these strategies, each owner's proposal offers the other owner a lower payoff than the latter obtains if the worker's proposal wins. Thus, neither owner has an incentive to deviate to the other's proposal (which would cause it to defeat the worker's proposal) or to any other proposal (which would have no effect), and in equilibrium the worker's proposal wins. Tax rate competition becomes tax base competition.

Finally, notice that the equilibrium national tax structure need not be symmetric. We have also shown that fiscal cooperation can occur with asymmetric national tax structures. We suspect that asymmetry is a necessary condition for fiscal cooperation to fail. It does not seem possible to show this analytically, however. For example, to prove this for the case in which the workers propose $0r0r$, we need to show not only $\hat{\mathcal{V}}_i^{0r0r} < \hat{\mathcal{V}}_i^{wrwr}$ and $G_i(w_i^{0r0r}, r_i^{0r0r}) > G_i(w_i^{wrwr}, r_i^{wrwr})$ for $i = A, B$, which we have done, but also $\hat{\mathcal{V}}_i^{0r0r} < \hat{\mathcal{V}}_i^{w0w0}$ and $G_i(w_i^{0r0r}, r_i^{0r0r}) > G_i(w_i^{w0w0}, r_i^{w0w0})$ for $i = A, B$. The proof of the first result relies on the fact that the permission to tax under $0r0r$ is strictly narrower than the permission to tax under $wrwr$. We cannot use this argument to show the second result. Showing the necessity of asymmetry for cooperation to fail when the workers propose other national tax structures will present similar problems.

7. Conclusions

We analyze the choice of tax bases and rates in a two-stage extensive form game. We find that as long as the owners in each region are not too aggressive, they will be able to shape tax structure in the first stage to reduce tax competition in the second stage. By agreeing on national tax structure in the first stage, the owners credibly commit to making both regions less attractive to workers. This makes both of them better off. We interpret this collusion as an agreement to reduce tax competition. Cooperation in the first stage also keeps the choice of tax structure out of the hands of the workers. When this occurs the outcome for the owners is poor, even though the owners still determine tax rates in the second stage. These results are somewhat in contrast to those emphasizing the role of prisoner's dilemmas in tax competition in the local public sector (Hoyt (1991), Nechyba (1997)).

Some extensions of the results and the model deserve further investigation. By focusing on analytical results and tax structure, we did not consider a number of issues that the model can address. Through simulations we could consider how the tax rates and amount of local public good are affected by fiscal cooperation. We could also consider the importance of asymmetry in tax structure in the breakdown of fiscal cooperation. Variations on the model could consider functional forms in which the local public good is important but not essential to all agents or local public infrastructure instead of goods. Perhaps the most interesting extension would include an earlier stage of voting on a move from autarkic regions to those with the economic and political structure developed here. Positive and normative questions about the likelihood and benefit of economic integration or decentralization require formal models with national polities that have important but limited influence on regional policy.

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Proof of Theorem 1.

Part i. If $(w_A, r_A, w_B, r_B) \in \mathcal{E}_A \times \mathcal{E}_B$ then any allocation of labor satisfying $N_i \geq 0$ and $\sum_{i=1}^{N} N_i = n$ is a migration equilibrium. The reason is that $G_A = G_B = 0$, so (6) and (7) immediately give $V_A = V_B = 0$ for all (N_A, N_B). In particular, (N_A, N_B) with $N_A = n^{00}$ and $N_B = n - n^{00}$, or $N_A = n^{01}$ and $N_B = n - n^{01}$, etc., are migration equilibria.

Now suppose $(w_A, r_A, w_B, r_B) \in [0,1]^4 \setminus \mathcal{E}_A \times \mathcal{E}_B$. We verify that (N_A, N_B) with

$$N_A = \frac{n\,[G_A(w_A, r_A)]^{\gamma}}{[G_A(w_A, r_A)]^{\gamma} + [G_B(w_B, r_B)]^{\gamma}}$$

and $N_B = n - N_A$ form a migration equilibrium. First, we always have $G_A(w_A, r_A) > 0$ or $G_B(w_B, r_B) > 0$ (or both), so the expressions are well defined. If $N_A > 0$ and $N_B > 0$ then using the formulas in (6) gives $V_A(G_A, N_A) = V_B(G_B, N_B)$. If $N_A > 0$ and $N_B = 0$ then $N_A = n$, the formula for N_A gives $G_B(w_B, r_B) = 0$, and the tax rates in A must then satisfy $G_A(w_A, r_A) > 0$. It then follows from (6) and (7) that $V_A(G_A, N_A) > 0 = V_B(G_B, N_B)$. The case $N_A = 0$ and $N_B > 0$ is similar.

Part ii. If $N_A > 0$ and $N_B > 0$ form a migration equilibrium then from the definition we must have $V_A(G_A, N_A) = V_B(G_B, N_B)$. This has a unique solution, given above.

If $N_A = n$ and $N_B = 0$ form a migration equilibrium then from the definition we must have $V_A(G_A, N_A) \geq V_B(G_B, N_B)$. By assumption we cannot have both $G_A = 0$ and $G_B = 0$. Suppose $G_A > 0$ and $G_B > 0$. Then $+\infty > V_A(G_A, N_A) \geq V_B(G_B, N_B) = +\infty$, a contradiction. Suppose $G_A = 0$ and $G_B > 0$. In this case $0 = V_A(G_A, N_A) \geq V_B(G_B, N_B) = +\infty$, a contradiction. Finally, suppose $G_A > 0$ and $G_B = 0$. If some other allocation (N'_A, N'_B) is a migration equilibrium, then we have $N'_B > 0$ and so $0 = V_B(G_B, N'_B) \geq$

$V_A(G_A, N'_A)$. Therefore $V_A(G_A, N'_A) = 0$, a contradiction whether $N'_A > 0$ (by (6)) or $N'_A = 0$ (by (7)).

Part *iii*. From part *i*, $N_i(\cdot)$ is clearly continuous on $[0,1]^4 \setminus \mathcal{E}_A \times \mathcal{E}_B$.

Now fix i and any point $(w_A, r_A, w_B, r_B) \in \mathcal{E}_A \times \mathcal{E}_B$. We give a complete demonstration since we use the constructions in later proofs.

Suppose first that $N_i(w_A, r_A, w_B, r_B) < n$. Every neighborhood of (w_i, r_i) contains a point (w'_i, r'_i) in $[0,1]^2 \setminus \mathcal{E}_i$. Therefore $G_i(w'_i, r'_i) > 0$, $G_{-i}(w_{-i}, r_{-i}) = 0$, and $N_i(w'_i, r'_i, w_{-i}, r_{-i}) = n$.[17] We can therefore construct a sequence $\left\{(w_i^k, r_i^k, w_{-i}^k, r_{-i}^k)\right\}_{k=1}^{\infty}$ that converges to $(w_i, r_i, w_{-i}, r_{-i})$ such that $N_i(w_i^k, r_i^k, w_{-i}^k, r_{-i}^k) = n$ at all terms. Therefore $\lim_{k\to\infty} N_i(w_i^k, r_i^k, w_{-i}^k, r_{-i}^k) = n > N_i(w_i, r_i, w_{-i}, r_{-i})$. This establishes the discontinuity.

Suppose instead $N_i(w_i, r_i, w_{-i}, r_{-i}) = n$. In this case there is still a discontinuity, but in (w_{-i}, r_{-i}) and not (w_i, r_i). Every neighborhood of (w_{-i}, r_{-i}) contains a point (w'_{-i}, r'_{-i}) in $[0,1]^2 \setminus \mathcal{E}_{-i}$. Therefore $G_i(w_i, r_i) = 0$, $G_{-i}(w'_{-i}, r'_{-i}) > 0$, and $N_i(w'_i, r'_i, w_{-i}, r_{-i}) = 0$. We can therefore construct a sequence $\left\{(w_i^k, r_i^k, w_{-i}^k, r_{-i}^k)\right\}_{k=1}^{\infty}$ that converges to $(w_i, r_i, w_{-i}, r_{-i})$ such that $N_i(w_i^k, r_i^k, w_{-i}^k, r_{-i}^k) = 0$ at all terms. Therefore $\lim_{k\to\infty} N_i(w_i^k, r_i^k, w_{-i}^k, r_{-i}^k) = 0 < n = N_i(w_i, r_i, w_{-i}, r_{-i})$. Again this establishes the discontinuity.

Proof of Theorem 2.

Part *i*. Fix any point (w_A, r_A, w_B, r_B) in $\mathcal{E}_A \times \mathcal{E}_B \setminus \overline{\mathcal{E}}$. As shown in Theorem 1, part *iii*, for both agents we can construct a sequence $\left\{(w_i^k, r_i^k, w_{-i}^k, r_{-i}^k)\right\}_{k=1}^{\infty}$ converging to $(w_i, r_i, w_{-i}, r_{-i})$

[17] Note that by convention, if $i = A$ then the vector $(w'_i, r'_i, w_{-i}, r_{-i})$ denotes (w'_A, r'_A, w_B, r_B) and if $i = B$ it denotes (w_A, r_A, w'_B, r'_B).

such that $\lim_{k\to\infty} N_i(w_i^k, r_i^k, w_{-i}^k, r_{-i}^k)$ exists and $\lim_{k\to\infty} N_i(w_i^k, r_i^k, w_{-i}^k, r_{-i}^k) = n \neq N_i(w_i, r_i, w_{-i}, r_{-i})$. We therefore have $\lim_{k\to\infty} \hat{\mathcal{V}}_i\ (w_i^k, r_i^k, w_{-i}^k, r_{-i}^k) = \lim_{k\to\infty}\ H_i(w_i^k, r_i^k)\ N_i(w_A^k, r_A^k, w_B^k, r_B^k) = H_i(w_i, r_i)n$ while $\hat{\mathcal{V}}_i(w_i, r_i, w_{-i}, r_{-i}) = H_i(w_i, r_i)N_i(w_i, r_i, w_{-i}, r_{-i})$. This holds for both $i = A, B$. The fact that (w_A, r_A, w_B, r_B) lies in $\mathcal{E}_A \times \mathcal{E}_B$ but not in $\overline{\mathcal{E}}$ implies that $H_i(w_i, r_i) > 0$ for at least one of A or B. Letting i denote this specific player, we can divide the previous expressions by $H_i(w_i, r_i) > 0$. It follows that $\lim_{k\to\infty} \hat{\mathcal{V}}_i(w_i^k, r_i^k, w_{-i}^k, r_{-i}^k) \neq \hat{\mathcal{V}}_i(w_i, r_i, w_{-i}, r_{-i})$. This establishes the discontinuity.

Part ii. Continuity on $[0,1]^4 \setminus (\mathcal{E}_A \times \mathcal{E}_B)$ is immediate from Theorem 1, part i. Choose any point $(w_A, r_A, w_B, r_B) \in \overline{\mathcal{E}}$. We have $H_i(w_i, r_i) = 0$ for both $i = A, B$. Therefore $\hat{\mathcal{V}}_i(w_A, r_A, w_B, r_B) = 0$. Let $\{(w_A^k, r_A^k, w_B^k, r_B^k)\}_{k=1}^{\infty}$ be any sequence of points converging to any point in $\overline{\mathcal{E}}$. At all terms we must have $H_i(w_i^k, r_i^k)n \geq H_i(w_i^k, r_i^k)\ N_i(w_A^k, r_A^k, w_B^k, r_B^k) \geq 0$. We know $\lim_{k\to\infty}\ H_i(w_i^k, r_i^k)n = 0$. It then follows by a standard theorem that $\lim_{k\to\infty}$ $H_i(w_i^k, r_i^k)\ N_i(w_A^k, r_A^k, w_B^k, r_B^k) = 0$. Therefore $\lim_{k\to\infty}\ \hat{\mathcal{V}}_i(w_A^k, r_A^k, w_B^k, r_B^k) = 0$. Since this holds for all sequences and both agents i, the conclusion follows.

Before establishing Theorem 3, we establish the basic properties of the model in a series of Lemmas.

Lemma 1.

i. $\ln H_i$ is strictly concave on $(0,1) \times (0,1)$.

ii. $\ln G_i$ is strictly concave on $(0,1) \times (0,1)$.

iii. Define the function $Z : \Re^2 \longrightarrow n$ by:

$$Z(x,y) = \frac{n\,[\exp(x)]^\gamma}{[\exp(x)]^\gamma + [\exp(y)]^\gamma}$$

Then $\ln Z$ is strictly concave in x on $(0,1)$.

iv. $N_i(w_A, r_A, w_B, r_B) = Z[\ln G_i(w_i, r_i), \ln G_{-i}(w_{-i}, r_{-i})]$, $i = A, B$.

v. $\ln\hat{\mathcal{V}}_i(\cdot, \cdot, w_{-i}, r_{-i})$ is strictly concave on $(0,1) \times (0,1)$.

vi. $\ln\hat{\mathcal{V}}_i(\cdot, 0, w_{-i}, r_{-i})$ and $\ln\hat{\mathcal{V}}_i(0, \cdot, w_{-i}, r_{-i})$ are each strictly concave on $(0,1)$.

Proof of Lemma 1.

Part *i*. Since h_i is a positive constant, the conclusion follows if and only if it holds for $H(w_i, r_i) = H_i(w_i, r_i)/h_i$. The derivatives of $\ln H(w_i, r_i)$ are:

$$\frac{\partial \ln H}{\partial w_i} = \frac{\beta}{\beta w_i + (1-\beta) r_i} \qquad \frac{\partial \ln H}{\partial r_i} = \frac{1-\beta}{\beta w_i + (1-\beta) r_i} - \frac{\alpha}{1-r_i}$$

$$\frac{\partial^2 \ln H}{\partial w_i^2} = \frac{-\beta^2}{[\beta w_i + (1-\beta) r_i]^2} \quad \frac{\partial^2 \ln H}{\partial r_i^2} = -\left\{\frac{(1-\beta)^2}{[\beta w_i + (1-\beta) r_i]^2} + \frac{\alpha}{(1-r_i)^2}\right\}$$

$$\frac{\partial^2 \ln H}{\partial w_i \partial r_i} = \frac{-\beta(1-\beta)}{[\beta w_i + (1-\beta) r_i]^2}$$

The Hessian is negative definite on $(0,1)^2$ since $\frac{\partial^2 \ln H}{\partial w_i^2} < 0$ and:

$$\frac{\partial^2 \ln H}{\partial w_i^2}\frac{\partial^2 \ln H}{\partial r_i^2} - \left(\frac{\partial^2 \ln H}{\partial w_i \partial r_i}\right)^2 = \frac{\alpha\beta^2}{[\beta w_i + (1-\beta) r_i]^2 (1-r_i)^2} > 0$$

Part ii. Since g_i is a positive constant, the conclusion follows if and only if it holds for $G(w_i, r_i) = G_i(w_i, r_i)/g_i$. The derivatives of $\ln G_i(w_i, r_i)$ are:

$$\frac{\partial \ln G}{\partial w_i} = \frac{\beta}{\beta w_i + (1-\beta) r_i} - \frac{\alpha}{1-w_i} \qquad \frac{\partial \ln G}{\partial r_i} = \frac{1-\beta}{\beta w_i + (1-\beta) r_i}$$

$$\frac{\partial^2 \ln G}{\partial w_i^2} = -\left\{ \frac{\beta^2}{[\beta w_i + (1-\beta) r_i]^2} + \frac{\alpha}{(1-w_i)^2} \right\} \qquad \frac{\partial^2 \ln G}{\partial r_i^2} = \frac{-(1-\beta)^2}{[\beta w_i + (1-\beta) r_i]^2}$$

$$\frac{\partial^2 \ln G}{\partial w_i \partial r_i} = \frac{-\beta(1-\beta)}{[\beta w_i + (1-\beta) r_i]^2}$$

The Hessian is negative definite on $(0,1)^2$ since $\frac{\partial^2 \ln G}{\partial w_i^2} < 0$ and

$$\frac{\partial^2 \ln G}{\partial w_i^2} \frac{\partial^2 \ln G}{\partial r_i^2} - \left(\frac{\partial^2 \ln G}{\partial w_i \partial r_i} \right)^2 = \frac{\alpha (1-\beta)^2}{[\beta w_i + (1-\beta) r_i]^2 (1-w_i)^2} > 0$$

Part iii. We have:

$$\frac{\partial \ln Z}{\partial x} = \frac{\gamma}{1 + \left[\frac{\exp(x)}{\exp(y)}\right]^\gamma} > 0$$

$$\frac{\partial^2 \ln Z}{\partial x^2} = -\gamma^2 \frac{\left[\frac{\exp(x)}{\exp(y)}\right]^\gamma}{\left\{1 + \left[\frac{\exp(x)}{\exp(y)}\right]^\gamma\right\}^2} < 0$$

Part iv. Substitute $\ln G_i$ and $\ln G_{-i}$ into the definition of Z and use the definition of N_i.

Part v. We have:

$$\begin{aligned} \ln \hat{\mathcal{V}}_i(w_A, r_A, w_B, r_B) &= \ln H_i(w_i, r_i) + \beta(\alpha+1) \ln N_i(w_A, r_A, w_B, r_B) \\ &= \ln H_i(w_i, r_i) + \beta(\alpha+1) \ln Z[\ln G_i(w_i, r_i), \ln G_{-i}(w_{-i}, r_{-i})] \end{aligned} \quad (14)$$

The first term on the right is concave by part i of this Lemma and the second term on the right is the composition of functions that are concave in (w_i, r_i) by parts ii and iii and therefore similarly concave. The sum of concave functions is concave, giving the result.

Part vi. Similar to v.

Lemma 2. Assume $G_{-i} > 0$.

i. $$\underset{(w_i, r_i) \in [0,1]^2}{\text{argmax}} \ \hat{\mathcal{V}}_i(w_i, r_i, G_{-i}) = \left(\hat{W}_i^{wr}(G_{-i}), \hat{R}_i^{wr}(G_{-i})\right)$$

where $(\hat{W}_i^{wr}(G_{-i}), \hat{R}_i^{wr}(G_{-i}))$ are the unique solutions to $\frac{\partial \ln \hat{\mathcal{V}}_i}{\partial w_i} = 0$ and $\frac{\partial \ln \hat{\mathcal{V}}_i}{\partial r_i} = 0$, and they lie in $(0, 1)^2$.

ii. $$\underset{w_i \in [0,1]}{\text{argmax}} \ \hat{\mathcal{V}}(w_i, 0, G_{-i}) = \hat{W}_i^{w}(G_{-i})$$

where $\hat{W}_i^{w}(G_{-i})$ is the unique solution to $\frac{\partial \ln \hat{\mathcal{V}}_i}{\partial w_i} = 0$ given $r_i = 0$, and it lies in $(0, 1)$.

iii. $$\underset{r_i \in [0,1]}{\text{argmax}} \ \hat{\mathcal{V}}_i(0, r_i, G_{-i}) = \hat{R}_i^{r}(G_{-i})$$

where $\hat{R}_i^{r}(G_{-i})$ is the unique solution to $\frac{\partial \ln \hat{\mathcal{V}}_i}{\partial r_i} = 0$ given $w_i = 0$, and it lies in $(0, 1)$.

Proof of Lemma 2.

Part i. Differentiating (14) with respect to w_i and r_i, setting the results to zero and rearranging (as well as using both equations together) gives:

$$\left(\frac{1}{\alpha\gamma}\right)\left(\frac{g_i}{1+\alpha}\right)^{\gamma}(1-w_i)^{\alpha\gamma+1} - w_i G_{-i}^{\gamma} = 0 \tag{15}$$

$$\beta w_i + (1-\beta) r_i = \frac{1}{1+\alpha} \tag{16}$$

Notice that equation (15) is independent of r_i. The left hand side is positive for w_i close enough to zero and negative for w_i close enough to 1, so there is a solution in the open interval $(0, 1)$. At this value, it follows from (16) that $\frac{1}{(1+\alpha)(1-\beta)} - \frac{\beta}{1-\beta} \leq r_A \leq \frac{1}{(1+\alpha)(1-\beta)}$. We have $0 < \frac{1}{(1+\alpha)(1-\beta)} - \frac{\beta}{1-\beta}$ by concavity (equation (5)) and $\frac{1}{(1+\alpha)(1-\beta)} < 1$ by stability (equation (8)). We conclude that a solution to the first order conditions exists and lies in $(0,1)^2$. Since $\ln\hat{\mathcal{V}}_i$ is strictly concave (Lemma 1 (v)), we know the solution is unique and defines the global maximum on $(0,1)^2$. It is also the unique global maximum of $\hat{\mathcal{V}}_i$ on this set. Continuity of $\hat{\mathcal{V}}_i$ then implies it is the unique global maximum on $[0,1]^2$.

Part ii. Differentiating (14) with respect to w_i, setting the result to zero and imposing the restriction $r_i = 0$ gives:

$$\left(\frac{1}{\alpha\gamma}\right)(\beta g_i)^\gamma w_i^\gamma (1-w_i)^{\alpha\gamma+1} + G_{-i}^\gamma \left\{1 - w_i[1+\beta(1+\alpha)]\right\} = 0 \tag{17}$$

As in the previous case, the left hand side is positive for w_i close enough to zero and negative for w_i close enough to 1, so there is a solution in the open interval $(0,1)$. The remainder of the argument is the same as above, using Lemma 1 (vi).

Part iii. Differentiating (14) with respect to r_i, setting the result to zero and imposing the restriction $w_i = 0$ gives:

$$\left(\frac{1}{\alpha\gamma}\right)[(1-\beta)g_i]^\gamma r_i^\gamma [1 - r_i(1+\alpha)] + G_{-i}^\gamma [1 - r_i(1+\alpha)(1-\beta)] = 0 \tag{18}$$

As in the previous case, the left hand side is positive for r_i close enough to zero and negative for r_i close enough to 1, so there is a solution in the open interval $(0,1)$. The remainder of the argument is the same as above, using Lemma 1 (vi).

Lemma 3. Assume $G_{-i} > 0$.

i. $\frac{\partial \hat{R}_i^{wr}}{\partial G_{-i}} > 0, \quad \frac{\partial \hat{W}_i^{wr}}{\partial G_{-i}} < 0$

ii. $\frac{\partial \hat{W}_i^{w}}{\partial G_{-i}} < 0$

iii. $\frac{\partial \hat{R}_i^{r}}{\partial G_{-i}} > 0$

Proof of Lemma 3.

Part *i*. Define $F_1(w_i, r_i, G_{-i})$ and $F_2(w_i, r_i, G_{-i})$ by the left hand sides of (15) and (16). Clearly $\frac{\partial F_1}{\partial w_i} < 0$ and $\frac{\partial F_1}{\partial r_i} = 0$, so:

$$\Delta \equiv \begin{vmatrix} \frac{\partial F_1}{\partial w_i} & \frac{\partial F_1}{\partial r_i} \\ \frac{\partial F_2}{\partial w_i} & \frac{\partial F_2}{\partial r_i} \end{vmatrix} = \begin{vmatrix} \frac{\partial F_1}{\partial w_i} & 0 \\ \beta & 1-\beta \end{vmatrix} < 0$$

Clearly $\frac{\partial F_1}{\partial G_{-i}} < 0$, so now:

$$\frac{\partial \hat{W}_i^{wr}}{\partial G_{-i}} = -\frac{\begin{vmatrix} \frac{\partial F_1}{\partial G_{-i}} & \frac{\partial F_1}{\partial r_i} \\ \frac{\partial F_2}{\partial G_{-i}} & \frac{\partial F_2}{\partial r_i} \end{vmatrix}}{\Delta} = -\frac{\begin{vmatrix} \frac{\partial F_1}{\partial G_{-i}} & 0 \\ 0 & 1-\beta \end{vmatrix}}{\Delta} < 0$$

$$\frac{\partial \hat{R}_i^{wr}}{\partial G_{-i}} = -\frac{\begin{vmatrix} \frac{\partial F_1}{\partial w_i} & \frac{\partial F_1}{\partial G_{-i}} \\ \frac{\partial F_2}{\partial w_i} & \frac{\partial F_2}{\partial G_{-i}} \end{vmatrix}}{\Delta} = -\frac{\begin{vmatrix} \frac{\partial F_1}{\partial w_i} & \frac{\partial F_1}{\partial G_{-i}} \\ \beta & 0 \end{vmatrix}}{\Delta} > 0$$

Part *ii*. Note first that at the value of w_i that solves (17), the latter implies $1 - w[1 + \beta(1+\alpha)] < 0$, and this with (8) gives $1 - w[\alpha + (1-\beta)(1+\alpha)] < 0$. Now define $F(w_i, r_i, G_{-i})$ by the left hand side of (17). Differentiating gives:

$$\frac{\partial F}{\partial G_{-i}} = \gamma G_{-i}^{\gamma-1} \left\{1 - w_i[1 + \beta(1+\alpha)]\right\}$$

$$\frac{\partial F}{\partial w_i} = \left(\frac{1}{\alpha\gamma}\right) (\beta g_i)^{\gamma} w_i^{\gamma-1} (1-w_i)^{\alpha\gamma} \gamma \left\{1 - w_i[\alpha + (1-\beta)(1+\alpha)]\right\} - G_{-i}^{\gamma}[1 + \beta(1+\alpha)]$$

Both derivatives are negative, and the result follows.

Part *iii*. Note first that at the value of r_i that solves (18), the latter implies:

$$\frac{1 - r_i(1+\alpha)}{1 - r_i(1+\alpha)(1-\beta)} < 0$$

Necessarily $r_i(1+\alpha) > r_i(1+\alpha)(1-\beta) > 0$, so $1 - r_i(1+\alpha) < 1 - r_i(1+\alpha)(1-\beta)$, and so with the previous result:

$$1 - r_i(1+\alpha) < 0 < 1 - r_i(1+\alpha)(1-\beta)$$

Now define $F(w_i, r_i, G_{-i})$ by the left hand side of (18). Differentiating gives:

$$\frac{\partial F}{\partial G_{-i}} = \gamma G_{-i}^{\gamma-1}[1 - r_i(1+\alpha)(1-\beta)]$$

$$\frac{\partial F}{\partial r_i} = \left(\frac{1}{\alpha\gamma}\right)[(1-\beta)g_i]^{\gamma} r_i^{\gamma-1}\left\{\gamma[1 - r_i(1+\alpha)] - r(1+\alpha)\right\} - G_{-i}^{\gamma}(1+\alpha)(1-\beta)$$

The derivatives are positive and negative, respectively, and the result follows.